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# One-Parameter Families of Feasible Sets in Semi-infinite Optimization

H.Th. JONGEN[1] and J-J. RÜCKMANN[2]
[1]*Aachen University of Technology, Dept. of Mathematics (C), D-52056 Aachen, Germany (email: jongen@rwth-aachen.de)*
[2]*Technische Universität München, Zentrum Mathematik, Lehrstuhl für Angewandte Mathematik, D-80290 München, Germany (email: rueckman@appl-math.tu-muenchen.de)*

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**Abstract.** Feasible sets in semi-infinite optimization are basically defined by means of infinitely many inequality constraints. We consider one-parameter families of such sets. In particular, all defining functions – including those defining the index set of the inequality constraints – will depend on a parameter. We note that a semi-infinite problem is a two-level problem in the sense that a point is feasible if and only if all *global* minimizers of a corresponding marginal function are nonnegative.

For a quite natural class of mappings we characterize changes in the global topological structure of the corresponding feasible set as the parameter varies. As long as the index set (-mapping) of the inequality constraints is lower semicontinuous, all changes in topology are those which generically appear in one-parameter sets defined by finitely many constraints. In the case, however, that some component of the mentioned index set is born (or vanishes), the topological change is of global nature and is not controllable. In fact, the change might be as drastic as that when adding or deleting an (arbitrary) inequality constraint.

**Key words:** Bifurcation point, feasible set, Mangasarian-Fromovitz constraint qualification, parametric family, semi-infinite optimization,

**AMS subject classification:** 90 C 31, 90 C 34, 49 M 39

## 1. Introduction

In this paper we consider feasible sets of semi-infinite optimization problems that depend on a real parameter $t \in \mathbb{R}$; *semi-infinite* means that these sets are subsets of a *finite*-dimensional space and the number of inequality constraints is *infinite*. Semi-infinite optimization became a very active research topic in the last two decades; for a recent survey we refer to the book [16] that contains several tutorials as well as overview articles on the theory, numerics and applications in semi-infinite optimization.

As a starting point of this paper, we consider the feasible sets

$$M(t) \subset \mathbb{R}^n$$

of a parameter dependent semi-infinite optimization problem

$$P(t) \qquad \text{Minimize } \varphi(x,t) \text{ subject to } x \in M(t),$$

where the parameter $t \in \mathbb{R}$ is varying in a specific set under consideration.

Several practical applications and mathematical techniques (e.g. homotopy methods) lead to parametric semi-infinite optimization problems; see, for example, the survey papers [5] and [15]. Recent work on parametric semi-infinite optimization is done in [4, 12, 13, 17].

In global optimization one is often interested in the sets

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \text{ is a global (respective local) minimizer of } P(t)\}$$

consisting of global (respective *all* local) minimizers of $P(t)$ as the parameter $t$ varies. In [4, 13] the set of minimizers of a one-parameter semi-infinite optimization problem has been studied extensively under generic conditions and it was shown that the appearance of singularities in the (one-parameter) set of minimizers is closely related to topological properties of the feasible sets $M(t)$.

In this paper we will investigate the topological structure of $M(t)$ and, for a quite natural class of constraints which describe $M(t)$, we will characterize possible topological changes in the structure of $M(t)$ as the parameter $t$ varies.

Let $\mathbb{R}^n$, $n \geqslant 1$ be $n$-dimensional space endowed with the Euclidean norm $\|\cdot\|$. For an open subset $\mathcal{O} \subseteq \mathbb{R}^n$ let $C^p(\mathcal{O}, \mathbb{R})$, $p \geqslant 1$ be the set of $p$-times continuously differentiable functions from $\mathcal{O}$ to $\mathbb{R}$. If confusion is excluded, we write $C^p$ instead of $C^p(\mathcal{O}, \mathbb{R})$. By $Df(\bar{x})$ we denote the derivative (row vector) of $f$ at $\bar{x} \in \mathcal{O}$ ($D_{x^1} f(\bar{x})$ denotes the vector of partial derivatives of $f$ with respect to the components of the subvector $x^1$ of $x$). For $f \in C^2(\mathcal{O}, \mathbb{R})$ the second derivatives $D^2 f(\bar{x})$, $D^2_{x^1 x^1} f(\bar{x}) \ldots$ are analogously defined.

The sets under consideration are of semi-infinite type and have the following standard form:

$$M^{(H,G,U,V)}(t) = \{x \in \mathbb{R}^n \mid h_i(x,t) = 0,\ i \in I,\ G(x,y,t) \geqslant 0,\ y \in Y^{(U,V)}(t)\}, \tag{1.1}$$

where

- $t \in \mathbb{R}$ is the parameter,
- $I = \{1, \ldots, m\}$, $m < n$, $H = (h_1, \ldots, h_m)$, $h_i \in C^p(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $i \in I$, $y \in \mathbb{R}^r$, $G \in C^p(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, \mathbb{R})$,
- $Y^{(U,V)}(t) = \{y \in \mathbb{R}^r \mid u_\ell(y,t) = 0,\ \ell \in A,\ v_k(y,t) \geqslant 0,\ k \in B\}$,
- $A = \{1, \ldots, a\}$, $a < r$, $U = (u_1, \ldots, u_a)$, $u_\ell \in C^p(\mathbb{R}^r \times \mathbb{R}, \mathbb{R})$, $\ell \in A$, $B = \{1, \ldots, b\}$, $V = (v_1, \ldots, v_b)$ and $v_k \in C^p(\mathbb{R}^r \times \mathbb{R}, \mathbb{R})$, $k \in B$.

Obviously, we have $\bar{x} \in M^{(H,G,U,V)}(\bar{t})$ if and only if

- $h_i(\bar{x}, \bar{t}) = 0$, $i \in I$, and
- all *global* minimizers of the so-called *lower level problem*

$$\text{Minimize } G(\bar{x}, y, \bar{t}) \text{ subject to } y \in Y^{(U,V)}(\bar{t})$$

are nonnegative.
Unless stated otherwise, all functions appearing in (1.1) will be once continuously differentiable. If we consider $M^{(H,G,U,V)}(t)$ and $Y^{(U,V)}(t)$ for a fixed vector function $(H, G, U, V)$ and $(U, V)$, we sometimes write $M(t)$ and $Y(t)$, respectively.

The goal of this paper is to investigate possible changes in the topological structure of $M^{(H,G,U,V)}(t)$ for increasing parameter $t$. Throughout the paper we will assume that the index set $Y^{(U,V)}(t)$ is compact. (Note that the standard form (1.1) also includes the case with finitely many inequality constraints of the type $G(x, y, t) \geqslant 0$.) The cardinality of $Y^{(U,V)}(t)$ might be infinite. Consequently, the set $M^{(H,G,U,V)}(t)$ is defined by means of a finite number of equality constraints and perhaps an *infinite* number of inequality constraints. In the case of a *finite* number of constraints, changes in the topological structure of the feasible set have been studied in [9]. The ideas from [9] may be applied to describe local changes of $Y^{(U,V)}(t)$ since the latter set is indeed defined by means of a finite number of constraints.

We will discuss the following questions:

- When does a change in the structure of $Y^{(U,V)}(t)$ induce a change in the structure of $M^{(H,G,U,V)}(t)$?
- When does the structure of $M^{(H,G,U,V)}(t)$ change although the structure of $Y^{(U,V)}(t)$ remains unchanged?
- Which changes in the topological structure of $M^{(H,G,U,V)}(t)$ can be classified, where $(H, G, U, V)$ is taken from a quite natural class of mappings?

The following example illustrates that a topological classification of a change in the structure of $M^{(H,G,U,V)}(t)$ is not always possible.

EXAMPLE 1.1. Assume that $r = 1$, $a = 0$, $b = 1$ and, hence, $Y(t) = \{y \in \mathbb{R} \mid v_1(y, t) \geqslant 0\}$. Moreover, let $Y_1 \subset \mathbb{R}$ be a compact interval, $\bar{y} \in \mathbb{R} \setminus Y_1$ and assume that $Y(t) = Y_1$ for $t < \bar{t}$, $Y(\bar{t}) = Y_1 \cup \{\bar{y}\}$ and let $Y(t)$ consist of two connected components for $t > \bar{t}$ (see Figure 1).

Consequently, as the parameter $t$ increases and passes the value $\bar{t}$, the additional constraint $G(x, \bar{y}, \bar{t}) \geqslant 0$ joins the description of $M(t)$ at $t = \bar{t}$; i.e. we have

$$M(t) = \begin{cases} M_1(t) & \text{if} \quad t < \bar{t} \\ M_1(\bar{t}) \cap M_2 & \text{if} \quad t = \bar{t}, \end{cases}$$

where $M_2 = \{x \in \mathbb{R}^n \mid G(x, \bar{y}, \bar{t}) \geqslant 0\}$ and

$$M_1(t) = \{x \in \mathbb{R}^n \mid h_i(x, t) = 0,\ i \in I,\ G(x, y, t) \geqslant 0,\ y \in Y(t)\}$$

(here, we do not specify the constraints $h_i(x, t) = 0$, $i \in I$ and $G(x, y, t) \geqslant 0$, $y \in Y(t)$). However, the structure of $M_2$ might be quite arbitrary and, hence, the change in the topological structure from $M(t)$, $t < \bar{t}$ to the intersection $M(\bar{t}) = M_1(\bar{t}) \cap M_2$ cannot be classified in general (cf. Figure 2).

As becomes clear later on, the (dis)appearance of components of $Y(t)$ cannot be excluded generically.

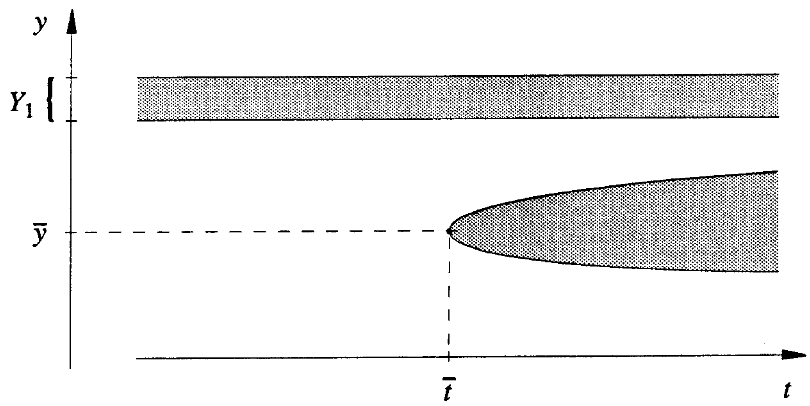


*Figure 1.*

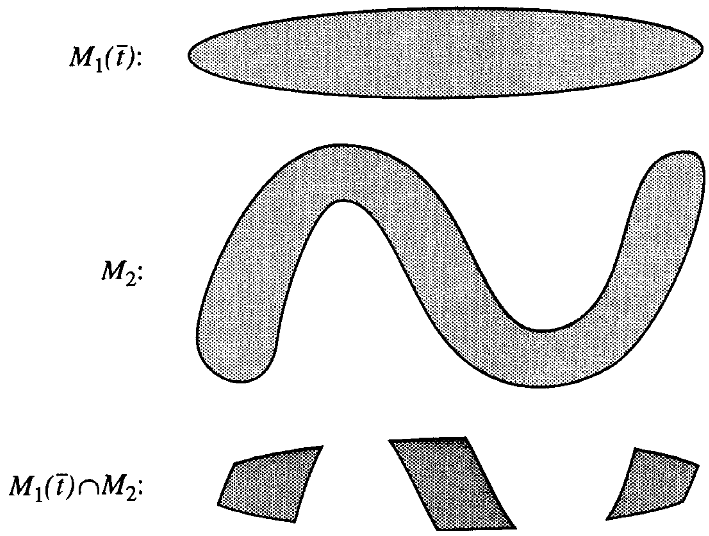


*Figure 2.*

The paper is organized as follows. Section 2 provides some notation and preliminary results. In Section 3 we summarize the main results from [9] on topological changes in the structure of $Y^{(U,V)}(t)$. Section 4 includes a genericity theorem on bifurcation points and the definition of appropriate subsets of the considered function spaces. Finally, in Section 5, classifications of topological changes in the structure of $M^{(H,G,U,V)}(t)$ are given, where $(H, G, U, V)$ is taken from an appropriate natural subset of mappings.

These classifications of a topological change in the structure of $M^{(H,G,U,V)}(t)$ locally around a considered parameter value $\bar{t}$ can be summarized in the following way:

- If the set-valued mapping $t \mapsto Y(t)$ is *not lower semicontinuous* at $\bar{t}$ (or, in other words, a new component of $\{(y,t) \mid y \in Y^{(U,V)}(t)\}$ is born locally), then the topological change in the structure of $M^{(H,G,U,V)}(t)$ might be arbitrarily drastic and cannot be classified in general (as illustrated in Example 1.1).
- If the set-valued mapping $t \mapsto Y(t)$ is *lower semicontinuous* at $\bar{t}$, then there are two possibilities:
  – Either $M^{(H,G,U,V)}(\bar{t})$ and $M^{(H,G,U,V)}(t)$ are homeomorphic for all $t$ near $\bar{t}$, i.e. the topological structure remains unchanged,
  – or the changes in the topological structure of $M^{(H,G,U,V)}(t)$ are those which also appear in one-parameter feasible sets of finite optimization problems (which are defined by finitely many equality and inequality constraints) and which are characterized in [9].

## 2. Notations, preliminary results

This section lists several notations and preliminary results which we will use later.

### Lower and upper semicontinuity

For $K \subset \mathbb{R}^r$, $\tilde{y} \in \mathbb{R}^r$ and $\gamma > 0$ let

$$
\begin{aligned}
&B_\gamma(\tilde{y}) = \{y \in \mathbb{R}^r \mid \|y - \tilde{y}\| < \gamma\},\\
&d(\tilde{y}, K) = \inf\{\|y - \tilde{y}\|,\ y \in K\} \quad \text{and}\\
&B_\gamma(K) = \{y \in \mathbb{R}^r \mid d(y, K) < \gamma\}.
\end{aligned}
$$

We recall the definitions of lower and upper semicontinuity of a set-valued mapping (cf. [1]).

The mapping $t \mapsto Y(t)$ is *lower semicontinuous* (briefly: *lsc*) *at* $\bar{t}$ if for any open set $\mathcal{O} \subset \mathbb{R}^r$ with $\mathcal{O} \cap Y(\bar{t}) \neq \emptyset$ there exists a neighbourhood $\mathcal{V}$ of $\bar{t}$ such that $\mathcal{O} \cap Y(t) \neq \emptyset$ whenever $t \in \mathcal{V}$.

The mapping $t \mapsto Y(t)$ is *upper semicontinuous* (briefly: *usc*) *at* $\bar{t}$ if for each $\gamma > 0$ there exists a neighbourhood $\mathcal{V}$ of $\bar{t}$ such that $Y(t) \subset B_\gamma(Y(\bar{t}))$ whenever $t \in \mathcal{V}$.

As a consequence of the latter definition it follows that $Y(\bar{t})$ is compact and the mapping $t \mapsto Y(t)$ is usc at $\bar{t}$ if and only if there exists a neighbourhood $\mathcal{V}$ of $\bar{t}$ and a compact set $\hat{Y} \subset \mathbb{R}^r$ such that $Y(t) \subset \hat{Y}$ for all $t \in \mathcal{V}$.

**Constraint qualifications**

The *Mangasarian–Fromovitz constraint qualification* (*MFCQ*) is said to hold at $\bar{y} \in Y^{(U,V)}(\bar{t})$ if the vectors $D_y u_\ell(\bar{y}, \bar{t})$, $\ell \in A$ are linearly independent and there exists a $w \in \mathbb{R}^r$ satisfying

$$D_y u_\ell(\bar{y}, \bar{t})w = 0, \quad \ell \in A$$
$$D_y v_k(\bar{y}, \bar{t})w > 0, \quad k \in B_0^{(V)}(\bar{y}, \bar{t}),$$

where $B_0^{(V)}(\bar{y}, \bar{t}) = \{k \in B \mid v_k(\bar{y}, \bar{t}) = 0\}$. Define

$$MF^{(U,V)}(\bar{t}) = \{y \in Y^{(U,V)}(\bar{t}) \mid \text{(MFCQ) holds at } y \in Y^{(U,V)}(\bar{t})\} \text{ and}$$
$$MF^{(U,V)} = \{(y,t) \in \mathbb{R}^r \times \mathbb{R} \mid y \in MF^{(U,V)}(t)\}.$$

The *Linear Independence constraint qualification* (*LICQ*) is said to hold at $\bar{y} \in Y^{(U,V)}(\bar{t})$ if the vectors $D_y u_\ell(\bar{y}, \bar{t})$, $\ell \in A$, $D_y v_k(\bar{y}, \bar{t})$, $k \in B_0^{(V)}(\bar{y}, \bar{t})$ are linearly independent.

The *Extended Mangasarian–Fromovitz constraint qualification* (*EMFCQ*) is said to hold at $\bar{x} \in H_{\bar{t}}^{-1}(0)$ if the vectors $D_x h_i(\bar{x}, \bar{t})$, $i \in I$ are linearly independent and there exists a $\xi \in \mathbb{R}^n$ satisfying

$$D_x h_i(\bar{x}, \bar{t})\xi = 0, \quad i \in I$$
$$D_x G(\bar{x}, y, \bar{t})\xi > 0, \quad y \in Y_0^{(G,U,V)}(\bar{x}, \bar{t}),$$

where $H_{\bar{t}}^{-1}(0) = \{x \in \mathbb{R}^n \mid h_i(x, \bar{t}) = 0,\ i \in I\}$ and
$Y_0^{(G,U,V)}(\bar{x}, \bar{t}) = \{y \in Y^{(U,V)}(\bar{t}) \mid G(\bar{x}, y, \bar{t}) = 0\}$.

The *Extended Linear Independence constraint qualification* (*ELICQ*) is said to hold at $\bar{x} \in M^{(H,G,U,V)}(\bar{t})$ if the vectors $D_x h_i(\bar{x}, \bar{t})$, $i \in I$, $D_x G(\bar{x}, y, \bar{t})$, $y \in Y_0^{(U,V,G)}(\bar{x}, \bar{t})$ are linearly independent.

Furthermore, define

$$M^{(H,G,U,V)} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in M^{(H,G,U,V)}(t)\},$$
$$H^{-1}(0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in H_t^{-1}(0)\} \quad \text{and}$$
$$Y^{(U,V)} = \{(y,t) \in \mathbb{R}^r \times \mathbb{R} \mid y \in Y^{(U,V)}(t)\}.$$

If no confusion is possible we will delete in the remainder of this paper the upper indices in $B_0^{(V)}(\bar{y}, \bar{t})$, $Y_0^{(U,V,G)}(\bar{x}, \bar{t})$, $M^{(H,G,U,V)}$ etc.

**The $C_s^p$-topology**

In Sections 3, 4 and 5 we will consider several subsets of the underlying function space that will be endowed with the strong $C^p$-topology, $p = 0, 1, 2, \ldots$ which is denoted by $C_s^p$ (for details see [7, 11]). For finite $p$ a $C_s^p$-neighbourhood of $f \in C^P(\mathbb{R}^n, \mathbb{R})$, $P \geqslant p$ in $C^P(\mathbb{R}^n, \mathbb{R})$ consists of all those functions $\tilde{f} \in C^P(\mathbb{R}^n, \mathbb{R})$ whose pointwise difference of the derivatives of $f$ and $\tilde{f}$ up to order

$p$ is controlled by a continuous positive function $\varepsilon(\cdot) : \mathbb{R}^n \to \mathbb{R}$. Hence, a typical $C_s^1$-base neighbourhood $\mathcal{P}^\varepsilon$ of the zero function in $C^P(\mathbb{R}^n, \mathbb{R})$, $P \geqslant 1$ is given by $\varepsilon : \mathbb{R}^n \to (0, \infty)$ as $\mathcal{P}^\varepsilon = \{\tilde{f} \in C^P(\mathbb{R}^n, \mathbb{R}) \mid |\tilde{f}(z)| + \sum_{i=1}^n |D_{x_i}\tilde{f}(z)| < \varepsilon(z)$ for all $z \in \mathbb{R}^n\}$. A typical $C_s^1$-base neighbourhood of $f \in C^P(\mathbb{R}^n, \mathbb{R})$ is $f + \mathcal{P}^\varepsilon$. The $C_s^p$-topology of the product space $C^P(\mathbb{R}^n, \mathbb{R}^q)$ $(= C^P(\mathbb{R}^n, \mathbb{R}) \times \cdots \times C^P(\mathbb{R}^n, \mathbb{R})$, $q$-times) is the induced product topology. The $C_s^\infty$-topology for $C^\infty(\mathbb{R}^n, \mathbb{R})$ is generated by means of the union of the bases for the $C_s^p$-topology, $p = 0, 1, 2, \ldots$.

The subsequent lemma follows directly from [3, Theorem B and Remark 3.4] and [8, Corollary 1].

LEMMA 2.1. *Let $\bar{t} \in \mathbb{R}$, $(\bar{U}, \bar{V}) \in C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ (i.e. $\bar{U} \in C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^a)$ and $\bar{V} \in C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^b)$) as well as $\delta : \mathbb{R}^r \to (0, \infty)$ be a given continuous positive function. Furthermore, assume that $Y^{(\bar{U},\bar{V})}(\bar{t})$ is a compact set with $Y^{(\bar{U},\bar{V})}(\bar{t}) = MF^{(\bar{U},\bar{V})}(\bar{t})$ and that the mapping $t \mapsto Y^{(\bar{U},\bar{V})}(t)$ is usc at $\bar{t}$. Then, there exists a $C_s^1$-neighbourhood $\vartheta$ of $(\bar{U}, \bar{V})$ in $C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ and a neighbourhood $\mathcal{V}$ of $\bar{t}$ as well as for each $(U, V) \in \vartheta$ and each $t \in \mathcal{V}$ a homeomorphism*

$$\phi^{(U,V,t)} : Y^{(\bar{U},\bar{V})}(\bar{t}) \longrightarrow Y^{(U,V)}(t)$$

*satisfying $\|\phi^{(U,V,t)}(y) - y\| < \delta(y)$ for all $y \in Y^{(\bar{U},\bar{V})}(\bar{t})$.*

LEMMA 2.2. [3, Lemma 2.4]

*Let $\bar{t} \in \mathbb{R}$, $(U, V) \in C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ and $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^r$ be disjoint closed subsets. Furthermore, let $(\tilde{U}, \tilde{V}) \in C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ belong to $F(\mathcal{C}_1, \mathcal{C}_2)$ if and only if*

(i) *$(U, V)$ und $(\tilde{U}, \tilde{V})$ coincide on $\mathcal{C}_1 \times \{\bar{t}\}$ and*

(ii) *(LICQ) holds at each $y \in Y^{(\tilde{U},\tilde{V})}(\bar{t}) \cap \mathcal{C}_2$.*

*Then, $F(\mathcal{C}_1, \mathcal{C}_2)$ intersects every $C_s^1$-neighbourhood of $(U, V)$.*

The following corollary is an easy consequence of the latter two lemmas.

COROLLARY 2.3. *Let $\gamma > 0$ and assume that $\bar{y} \in MF^{(\bar{U},\bar{V})}(\bar{t})$. Then, there exist a neighbourhood $\mathcal{V}$ of $\bar{t}$ and a $C_s^1$-neighbourhood $\vartheta$ of $(\bar{U}, \bar{V})$ in $C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ such that for each $t \in \mathcal{V}$ and each $(U, V) \in \vartheta$ there is a $y(U, V, t) \in MF^{(U,V)}(t)$ satisfying $\|y(U, V, t) - \bar{y}\| < \gamma$.*

**The Reduction Ansatz**

In the remainder of Section 2 we assume that all appearing functions belong to $C^2$. Let $(H, G, U, V)$ be fixed. We recall the so-called Reduction Ansatz at a point $\bar{x} \in M(\bar{t})$ (cf. [6, 14, 2] for details).

Obviously, each $\bar{y} \in Y_0(\bar{x}, \bar{t})$ is a global minimizer of $G(\bar{x}, \cdot, \bar{t})|_{Y(\bar{t})}$; thus, there exist reals $\alpha_0 \geqslant 0$, $\alpha_\ell$, $\ell \in A$, $\beta_k \geqslant 0$, $k \in B_0(\bar{y}, \bar{t})$—not all vanishing—such that

$$\alpha_0 D_y G(\bar{x}, \bar{y}, \bar{t}) - \sum_{\ell \in A} \alpha_\ell D_y u_\ell(\bar{y}, \bar{t}) - \sum_{k \in B_0(\bar{y}, \bar{t})} \beta_k D_y v_k(\bar{y}, \bar{t}) = 0. \tag{2.1}$$

In particular, $\bar{y} \in Y_0(\bar{x}, \bar{t})$ is called a *nondegenerate* minimizer if the following three conditions are fulfilled:

- (LICQ) holds at $\bar{y} \in Y(\bar{t})$.
  Then, after fixing $\alpha_0 = 1$ the reals $\alpha_\ell = \bar{\alpha}_\ell$, $\ell \in A$, $\beta_k = \bar{\beta}_k$, $k \in B_0(\bar{y}, \bar{t})$ in (2.1) are uniquely determined;
- $\bar{\beta}_k \neq 0$, $k \in B_0(\bar{y}, \bar{t})$, and
- the matrix $\begin{pmatrix} L(\bar{y}, \bar{t}) & E(\bar{y}, \bar{t})^T \\ E(\bar{y}, \bar{t}) & 0 \end{pmatrix}$ is nonsingular, where

$$L(\bar{y}, \bar{t}) = D_y^2 G(\bar{x}, \bar{y}, \bar{t}) - \sum_{\ell \in A} \bar{\alpha}_\ell D_y^2 u_\ell(\bar{y}, \bar{t}) - \sum_{k \in B_0(\bar{y}, \bar{t})} \bar{\beta}_k D_y^2 v_k(\bar{y}, \bar{t})$$

  and the rows of $E(\bar{y}, \bar{t})$ are the derivatives of the active constraints at $\bar{y} \in Y(\bar{t})$:

$$E(\bar{y}, \bar{t}) = \begin{pmatrix} \vdots \\ D_y u_\ell(\bar{y}, \bar{t}), \ \ell \in A \\ \vdots \\ D_y v_k(\bar{y}, \bar{t}), \ k \in B_0(\bar{y}, \bar{t}) \\ \vdots \end{pmatrix}.$$

Now, we consider the mapping

$$T : \mathbb{R}^r \times \mathbb{R}^a \times \mathbb{R}^{|B_0(\bar{y}, \bar{t})|} \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^r \times \mathbb{R}^a \times \mathbb{R}^{|B_0(\bar{y}, \bar{t})|},$$

(where $|\cdot|$ denotes the cardinality) defined as

$$T(y, \alpha, \beta, x, t) = \begin{pmatrix} D_y G(x, y, t) - \sum\limits_{\ell \in A} \alpha_\ell D_y u_\ell(y, t) - \sum\limits_{k \in B_0(\bar{y}, \bar{t})} \beta_k D_y v_k(y, t) \\ u_\ell(y, t), \ \ell \in A \\ v_k(y, t), \ k \in B_0(y, t) \end{pmatrix},$$

where $\alpha = (\alpha_\ell, \ \ell \in A)$, $\beta = (\beta_k, \ k \in B_0(\bar{y}, \bar{t}))$. Obviously, we have $T(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{x}, \bar{t}) = 0$ and $D_{(y,\alpha,\beta)} T(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{x}, \bar{t})$ is nonsingular. By the Implicit Function Theorem, there exist neighbourhoods $\mathcal{O}_1$ of $\bar{x}$ and $\mathcal{O}_2$ of $\bar{t}$ as well as uniquely determined $C^1$-functions $(\tilde{y}(x, t), \tilde{\alpha}(x, t), \tilde{\beta}(x, t))$ on $\mathcal{O}_1 \times \mathcal{O}_2$ satisfying

$$\left\{ \begin{array}{l} (\tilde{y}(\bar{x}, \bar{t}), \tilde{\alpha}(\bar{x}, \bar{t}), \tilde{\beta}(\bar{x}, \bar{t})) = (\bar{y}, \bar{\alpha}, \bar{\beta}) \quad \text{and} \\ T(\tilde{y}(x, t), \tilde{\alpha}(x, t), \tilde{\beta}(x, t), x, t) = 0 \quad \text{for all} \quad (x, t) \in \mathcal{O}_1 \times \mathcal{O}_2. \end{array} \right. \tag{2.2}$$

Furthermore, the construction yields for $g(x,t) = G(x, \tilde{y}(x,t), t)$ that $g \in C^2(\mathcal{O}_1 \times \mathcal{O}_2, \mathbb{R})$.

The *Reduction Ansatz* (*RA*) is said to be *applicable at* $\bar{x} \in M(\bar{t})$ if each $\bar{y} \in Y_0(\bar{x}, \bar{t})$ is a nondegenerate minimizer.

Now, let $\bar{t} \in \mathbb{R}$ be fixed and assume that $Y(\bar{t})$ is compact, the mapping $t \mapsto Y(t)$ is usc at $\bar{t}$ and (RA) is applicable at $\bar{x} \in M(\bar{t})$. Then, the sets $M(t)$, $t$ near $\bar{t}$ can be described locally around $\bar{x}$ by *finitely many $C^2$-constraints*; indeed, the set $Y_0(\bar{x}, \bar{t})$ is finite and—for $Y_0(\bar{x}, \bar{t}) = \{\bar{y}^1, \ldots, \bar{y}^q\}$—there exist neighbourhoods $\mathcal{O}_1$ of $\bar{x}$ and $\mathcal{O}_2$ of $\bar{t}$ as well as functions

$$\tilde{y}^j : (x,t) \in \mathcal{O}_1 \times \mathcal{O}_2 \mapsto \tilde{y}^j(x,t) \in R^r, \quad j = 1, \ldots, q \tag{2.3}$$

defined analogously to $\tilde{y}(\bar{x}, \bar{t})$ in (2.2) such that for all $t \in \mathcal{O}_2$:

$$M(t) \cap \mathcal{O}_1 = \{x \in \mathcal{O}_1 \mid h_i(x,t) = 0,\ i \in I,\ g^j(x,t) \geqslant 0,\ j = 1, \ldots, q\}, \tag{2.4}$$

where $g^j(x,t) = G(x, \tilde{y}^j(x,t), t)$, $j = 1, \ldots, q$.

**Bifurcation points**

As we will see in the following sections a change in the topological structure of $Y^{(U,V)}(t)$ and $M^{(H,G,U,V)}(t)$ is closely related with the violation of (MFCQ) and (EMFCQ), respectively.

DEFINITION 2.4.

(i) A point $(\bar{y}, \bar{t}) \in Y^{(U,V)}$ is called a *bifurcation point of* $Y^{(U,V)}$, if $(\bar{y}, \bar{t}) \notin MF^{(U,V)}$. Let $Y_{bp}^{(U,V)}$ denote the set of bifurcation points of $Y^{(U,V)}$ and $Y_{bp}^{(U,V)}(\bar{t}) = \{y \in Y^{(U,V)}(\bar{t}) \mid (y, \bar{t}) \in Y_{bp}^{(U,V)}\}$ for $\bar{t} \in \mathbb{R}$.

(ii) $(\bar{y}, \bar{t}) \in Y_{bp}^{(U,V)}$ is called *nondegenerate* if the following three conditions are satisfied:

- The set $\{Du_\ell(\bar{y}, \bar{t}),\ \ell \in A,\ Dv_k(\bar{y}, \bar{t}),\ k \in B_0^{(V)}(\bar{y}, \bar{t})\}$ is linearly independent. In that case there exist reals $\hat{\alpha}_\ell$, $\ell \in A$, $\hat{\beta}_k \geqslant 0$, $k \in B_0^{(V)}(\bar{y}, \bar{t})$—unique up to a common multiple and not all vanishing—such that:

$$\sum_{\ell \in A} \hat{\alpha}_\ell D_y u_\ell(\bar{y}, \bar{t}) + \sum_{k \in B_0^{(V)}(\bar{y}, \bar{t})} \hat{\beta}_k D_y v_k(\bar{y}, \bar{t}) = 0;$$

- $\hat{\beta}_k > 0$, $k \in B_0^{(V)}(\bar{y}, \bar{t})$ and
- $W_1(\bar{y}, \bar{t})^T D_y^2 \mathcal{L}_1(\bar{y}, \bar{t}) W_1(\bar{y}, \bar{t})$ is nonsingular, where

$$\mathcal{L}_1(\bar{y}, \bar{t}) = \sum_{\ell \in A} \hat{\alpha}_\ell u_\ell(\bar{y}, \bar{t}) + \sum_{k \in B_0^{(V)}(\bar{y}, \bar{t})} \hat{\beta}_k v_k(\bar{y}, \bar{t})$$

and $W_1(\bar{y}, \bar{t})$ is a

$(r, r+1-a-|B_0^{(V)}(\bar{y}, \bar{t}\,)|)$-matrix whose columns form a basis of

$$\{y \in \mathbb{R}^r \mid D_y u_\ell(\bar{y}, \bar{t}\,)y = 0,\ \ell \in A,\ D_y v_k(\bar{y}, \bar{t}\,)y = 0,\ k \in B_0^{(V)}(\bar{y}, \bar{t}\,)\}.$$

DEFINITION 2.5. Let $Y^{(U,V)}(\bar{t}\,)$ be a compact set and the mapping $t \mapsto Y^{(U,V)}(t)$ be usc at $\bar{t}$.

(i) A point $(\bar{x}, \bar{t}\,) \in M^{(H,G,U,V)}$ is called a *bifurcation point of* $M^{(H,G,U,V)}$, if (EMFCQ) does not hold at $\bar{x} \in M^{(H,G,U,V)}(\bar{t}\,)$. Let $M_{bp}^{(H,G,U,V)}$ denote the set of bifurcation points of $M^{(H,G,U,V)}$ and $M_{bp}^{(H,G,U,V)}(\bar{t}\,) = \{x \in M^{(H,G,U,V)}(\bar{t}\,) \mid (x, \bar{t}\,) \in M_{bp}^{(H,G,U,V)}\}$ for $\bar{t} \in \mathbb{R}$.

(ii) $(\bar{x}, \bar{t}\,) \in M_{bp}^{(H,G,U,V)}$ is called *nondegenerate* if the following four conditions are satisfied:

- The set $\{Dh_i(\bar{x}, \bar{t}\,),\ i \in I,\ D_{(x,t)}G(\bar{x}, y, \bar{t}\,),\ y \in Y_0^{(U,V)}(\bar{x}, \bar{t}\,)\}$ is linearly independent. In that case the set $Y_0^{(U,V)}(\bar{x}, \bar{t}\,)$ is finite and for $Y_0^{(U,V)}(\bar{x}, \bar{t}\,) = \{\bar{y}^1, \ldots, \bar{y}^q\}$ there exist reals $\hat{\lambda}_i$, $i \in I$, $\hat{\mu}_j \geqslant 0$, $j = 1, \ldots, q$—unique up to a common multiple and not all vanishing—such that:

$$\sum_{i \in I} \hat{\lambda}_i D_x h_i(\bar{x}, \bar{t}\,) + \sum_{j=1}^{q} \hat{\mu}_j D_x G(\bar{x}, \bar{y}^j, \bar{t}\,) = 0;$$

- $\hat{\mu}_j > 0$, $j = 1, \ldots, q$;
- (RA) is applicable at $\bar{x} \in M^{(H,G,U,V)}(\bar{t}\,)$ and the corresponding functions $\tilde{y}^j$ and $g^j$, $j = 1, \ldots, q$ are defined as in (2.3) and (2.4), respectively, as well as
- $W_2(\bar{x}, \bar{t}\,)^T D_x^2 \mathcal{L}_2(\bar{x}, \bar{t}\,) W_2(\bar{x}, \bar{t}\,)$ is nonsingular, where

$$\mathcal{L}_2(\bar{x}, \bar{t}\,) = \sum_{i \in I} \hat{\lambda}_i h_i(\bar{x}, \bar{t}\,) + \sum_{j=1}^{q} \hat{\mu}_j g^j(\bar{x}, \bar{t}\,)$$

and $W_2(\bar{x}, \bar{t}\,)$ is an $(n, n+1-m-q)$-matrix whose columns form a basis of

$$\{x \in \mathbb{R}^n \mid D_x h_i(\bar{x}, \bar{t}\,)x = 0,\ i \in I,\ D_x G(\bar{x}, \bar{y}^j, \bar{t}\,)x = 0,\ j = 1, \ldots, q\}.$$

Note that $D_x g^j(\bar{x}, \bar{t}\,) = D_x G(\bar{x}, \bar{y}^j, \bar{t}\,)$, $j = 1, \ldots, q$.

## 3. Deformation of $Y(t)$

This section surveys several results from [9] about changes in the topological structure of $Y^{(U,V)}(t)$ when a nondegenerate point $(\bar{y}, \bar{t}\,) \in Y_{bp}^{(U,V)}$ appears.

REMARK 3.1. In [9] the class of bifurcation points is slightly broader than that used here. In fact, all points at which (LICQ) is violated are considered in [9]. However, if (LICQ) is violated but (MFCQ) is satisfied, then those points do not change the topological (i.e. homeomorphy) type of the set, see for example Lemma 2.1. Therefore, we do not take such points explicitly into account.

We start with the following lemma which follows from [9, Lemma 1.1].

LEMMA 3.2. *The set*

$$\mathcal{F}_1 = \{(U, V) \in C^2(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b}) \mid \textit{each } (y, t) \in Y_{bp}^{(U,V)} \textit{ is nondegenerate}\}$$

*is $C_s^2$-open/dense in $C^2$.*

If $(U, V) \in \mathcal{F}_1$, then the set $Y_{bp}^{(U,V)}$ of bifurcation points is a discrete set. Throughout this section let $(U, V) \in C^2$ be fixed.

**Types**

Let $(\bar{y}, \bar{t}) \in Y_{bp}$ be nondegenerate with $|B_0(\bar{y}, \bar{t})| = b_0$ and let the notations be chosen as in Definition 2.4. Then, $(\bar{y}, \bar{t})$ is one of the following two types.
*Type* 1: $b_0 = 0$.

$$\begin{aligned} \text{Type-numbers:}\ \delta_1 &= r + 1 - a \\ \delta_2 &= \text{number of positive eigenvalues of} \\ &\quad D_t\mathcal{L}_1(\bar{y}, \bar{t})W_1(\bar{y}, \bar{t})^T D_y^2\mathcal{L}_1(\bar{y}, \bar{t})W_1(\bar{y}, \bar{t}). \end{aligned}$$

(Obviously, we have $D_t\mathcal{L}_1(\bar{y}, \bar{t}) \neq 0$.)
Subtype 1a: $\delta_1 = \delta_2$ or $\delta_2 = 0$.
Subtype 1b: $\delta_1 \neq \delta_2$ and $\delta_2 \neq 0$.
*Type* 2: $b_0 \geqslant 1$.

$$\begin{aligned} \text{Type-numbers:}\ \delta_1 &= r + 1 - a - b_0 \\ \delta_2 &= \text{number of positive eigenvalues of} \\ &\quad W_1(\bar{y}, \bar{t})D_y^2\mathcal{L}_1(\bar{y}, \bar{t})W_1(\bar{y}, \bar{t}) \\ \delta_3 &= b_0 - 1 \\ \delta_4 &= \text{sign}\ D_t\mathcal{L}_1(\bar{y}, \bar{t}). \end{aligned}$$

Subtype 2a: $\delta_2 = 0$.
Subtype 2b: $\delta_2 \neq 0$.

**Topological changes**

Assume in the remainder of this section that $\hat{Y} \subset \mathbb{R}^r$ is a compact subset, $t_1 < t_2$ and $Y(t) \subset \hat{Y}$ for all $t \in [t_1, t_2]$. We will describe possible changes in the topological structure of $Y(t)$ as $t$ varies from $t_1$ to $t_2$. Here, we assume that $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2])$ is either empty or a singleton $\{(\bar{y}, \bar{t})\}$ with $\bar{t} \in (t_1, t_2)$ and $(\bar{y}, \bar{t})$ nondegenerate. For the used topological concepts we refer to [10, 18].

Let $D^k$ and $S^k$ denote a homeomorphic image of $\{z \in \mathbb{R}^k \mid \|z\| \leqslant 1\}$ and $\{z \in \mathbb{R}^{k+1} \mid \|z\| = 1\}$, respectively, where $S^{-1} = \emptyset$. Furthermore, let $\mathcal{M}_1$ be a $(k + \ell)$-dimensional manifold with boundary $\partial\mathcal{M}_1$ and $S^k \times D^\ell$ be embedded in

$\mathcal{M}_1 \setminus \partial\mathcal{M}_1$. We obtain a new manifold $\mathcal{M}_2$ by deleting $S^k \times D^\ell$ from $\mathcal{M}_1$ and put $D^{k+1} \times S^{\ell-1}$ in its place via homeomorphisms $\partial D^{k+1} \to S^k$ and $S^{\ell-1} \to \partial D^\ell$. If $\mathcal{N}_1$, $\mathcal{N}_2$ are manifolds homeomorphic with $\mathcal{M}_1$, $\mathcal{M}_2$, we say that $\mathcal{N}_2$ is obtained from $\mathcal{N}_1$ by deleting $S^k \times D^\ell$ and implanting $D^{k+1} \times S^{\ell-1}$.
The following theorem describes the possible topological changes (cf. Remark 3.1 and [9, Theorems 5.1, 5.2 and 5.3]).

THEOREM 3.3.
(i) *Let $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2])$ be empty. Then, $Y(t_1) \simeq Y(t_2)$ (where '$\simeq$' means 'is homeomorphic with').*

(ii) *Suppose that $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]) = \{(\bar{y}, \bar{t})\}$ with $\bar{t} \in (t_1, t_2)$ and $(\bar{y}, \bar{t})$ nondegenerate.*

*If $(\bar{y}, \bar{t})$ is of Type* 1 $(\delta_1, \delta_2)$ *then $Y(t_2)$ is obtained from $Y(t_1)$ by deleting $S^{\delta_2-1} \times D^{\delta_1-\delta_2}$ and implanting $D^{\delta_2} \times S^{\delta_1-\delta_2-1}$.*

*If $(\bar{y}, \bar{t})$ is of Type* 2 $(\delta_1, \delta_2, \delta_3, 1)$ *then $Y(t_2)$ is homotopy-equivalent to $Y(t_1)$ with $D^{\delta_2}$ attached.*

*If $(\bar{y}, \bar{t})$ is of Type* 2 $(\delta_1, \delta_2, \delta_3, -1)$ *then $Y(t_1)$ is homotopy-equivalent to $Y(t_2)$ with $D^{\delta_2}$ attached.*

REMARK 3.4. Note that Theorem 3.3(i) is an immediate consequence of Lemma 2.1. As an illustration of the topological changes described in Theorem 3.3(ii) we consider the case that $u_\ell, v_k \in C^\infty$. Then, there exists a smooth local coordinate transformation $Q$ of the form

$$Q : (y, t) \longrightarrow (Q_1(y, t), Q_2(t)), \;\; DQ_2(t) > 0$$

such that $Y$ can locally be described around $(\bar{y}, \bar{t}) \in Y_{bp}$ as follows (cf. [9]):

- $(\bar{y}, \bar{t})$ is of Type 1 $(\delta_1, \delta_2)$:

$$t = -\sum_{\nu=1}^{\delta_2} y_\nu^2 + \sum_{\nu=\delta_2+1}^{\delta_1} y_\nu^2.$$

- $(\bar{y}, \bar{t})$ is of Type 2 $(\delta_1, \delta_2, \delta_3, \delta_4)$:

$$\delta_4 t \geqslant -\sum_{\nu=1}^{\delta_2} y_\nu^2 + \sum_{\nu=\delta_2+1}^{\delta_1} y_\nu^2 + \sum_{\nu=\delta_1+1}^{\delta_1+\delta_3} y_\nu, \quad y_\nu \geqslant 0, \;\; \nu = \delta_1+1, \ldots, \delta_1+\delta_3.$$

The construction yields that a nondegenerate point $(\bar{y}, \bar{t}) \in Y_{bp}$ of Subtype 1a or of Subtype 2a is a local minimizer (local maximizer) of $\Phi(y, t)|_Y$ with $\Phi(y, t) = t$; then, a connected component of $Y$ is created locally around $(\bar{y}, \bar{t})$ for increasing (decreasing) values of $t$. We have seen in Example 1.1 that in this situation the topological structure of $M(t)$ might change drastically and almost arbitrarily.

If a nondegenerate point $(\bar{y}, \bar{t}) \in Y_{bp}$ is of Subtype 1b or 2b, then $(\bar{y}, \bar{t})$ is neither a local minimizer nor a local maximizer of $\Phi(y, t)|_Y$. In particular, the following lemma holds.

LEMMA 3.5. *Let $\bar{t} \in \mathbb{R}$ and assume that each $(\bar{y}, \bar{t}) \in Y_{bp}$ is nondegenerate and of Subtype 1b or 2b. Then, the set-valued mapping $t \mapsto Y(t)$ is lsc at $\bar{t}$.*

The latter lemma does not assume the compactness of $Y(\bar{t})$. The proof of this lemma follows for $u_\ell, v_k \in C^\infty$, $\ell \in A$, $k \in B$ from Lemma 2.1 and Remark 3.4 and in the considered general case from the additional fact that $C^\infty$ is $C^1_s$-dense in $C^1$ (cf. [7]).

## 4. The Genericity Theorem and the sets CUSC and BAP

In this section we show that nondegeneracy of bifurcation points is a *generic* property. Recall that a subset of a Baire space is called generic if it contains the intersection of countably many open and dense subsets. In particular, generic sets are dense (cf. [7, 11]).
Furthermore, we define the natural subsets CUSC and BAP which will play a crucial role when considering deformations of $M^{(H,G,U,V)}(t)$.

### The Genericity Theorem

THEOREM 4.1. (Genericity Theorem).
*Let $C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^m) \times C^\infty(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, \mathbb{R}) \times C^\infty(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$ be endowed with the $C^\infty_s$-topology. Then, the set*

$$\mathcal{F}_2 = \left\{(H, G, U, V) \in C^\infty \;\middle|\; \begin{array}{l} \text{each } (\bar{x}, \bar{t}) \in M_{bp}^{(H,G,U,V)} \text{ and each} \\ (\bar{y}, \bar{t}) \in Y_{bp}^{(U,V)} \text{ is nondegenerate} \end{array} \right\}$$

*is generic. In particular, $\mathcal{F}_2$ is $C^\infty_s$-dense.*

*Proof.* The $C^\infty_s$-density of $\mathcal{F}_2$ follows from the fact that $C^\infty$ endowed with the $C^\infty_s$-topology is a Baire space. Recall (cf. [11]) that the set $\{\mathfrak{A} \in \mathfrak{M}(k, \ell) \mid \operatorname{rank}(\mathfrak{A}) = d\}$ is a $C^\infty$-manifold with codimension $(k-d)(\ell-d)$ for $d = 0, \dots, \min\{k, \ell\}$; here $\mathfrak{M}(k, \ell)$ denotes the space of all real $(k, \ell)$-matrices. We will restrict ourselves to a sketch of the proof which uses the *Multi-jet Transversality Theorem* (cf. [11, Chapter 7]). Let $q \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$) and let $B^j \subset B$ with $|B^j| = b_j$, $j = 1, \dots, q$ be arbitrarily chosen and consider the following (multi) 1-jet extension:

$$(x^1, y^1, t^1, x^2, y^2, t^2, \dots, x^q, y^q, t^q) \mapsto$$

$$\begin{cases} (x^1, y^1, t^1, \underbrace{\dots, u_\ell(y^1, t^1), Du_\ell(y^1, t^1), \dots,}_{\ell \in A} \underbrace{\dots, v_k(y^1, t^1), Dv_k(y^1, t^1), \dots,}_{k \in B} \dots \\ \underbrace{\dots, h_i(x^1, t^1), Dh_i(x^1, t^1), \dots,}_{i \in I} G(x^1, y^1, t^1), DG(x^1, y^1, t^1), \end{cases}$$

$$\vdots$$

$$\Big\{ (x^q, y^q, t^q), \dots\dots\dots, G(x^q, y^q, t^q), DG(x^q, y^q, t^q)).$$

Let us now focus our attention to the points $(x^1, t^1) \in M_{bp}^{(H,G,U,V)}$ satisfying $|Y_0^{(G,U,V)}(x^1, t^1)| = q$ with $Y_0^{(G,U,V)}(x^1, t^1) = \{y^1, \ldots, y^q\}$, $B_0^{(V)}(y^j, t^j) = B^j$, $j = 1, \ldots, q$. Then the following system of equations has to be satisfied:

- $\underline{y^j \in Y^{(U,V)}(t^j),\ j = 1, \ldots, q:}$
  $u_\ell(y^j, t^j) = 0,\ \ell \in A,\quad v_k(y^j, t^j) = 0,\ k \in B^j.$
  Number of equations: $qa + \sum_{j=1}^{q} b^j$.
- $\underline{y^j \in Y_0^{(G,U,V)}(x^j, t^j),\ j = 1, \ldots, q:}$
  $$\operatorname{rank}\begin{pmatrix} D_y G(x^j, y^j, t^j) \\ D_y u_\ell(y^j, t^j),\ \ell \in A \\ D_y v_k(y^j, t^j),\ k \in B^j \end{pmatrix} \leqslant a + b^j.$$
  Minimal number of equations, i.e. for $\operatorname{rank}(\ldots) = a + b^j$: $qr - qa - \sum_{j=1}^{q} b^j$.
- Coupling equations: $\underline{x^j = x^{j+1},\ t^j = t^{j+1},\ j = 1, \ldots, q-1:}$
  Number of equations: $(n+1)(q-1) = nq + q - n - 1$.
- $\underline{x^1 \in M^{(H,G,U,V)}(t^1):}$
  $h_i(x^1, t^1) = 0,\ i \in I,\quad G(x^1, y^j, t^1) = 0,\ j = 1, \ldots, q.$
  Number of equations: $m + q$.
- $\underline{\text{(ELICQ) does not hold at } x^1 \in M^{(H,G,U,V)}(t^1):}$
  $$\operatorname{rank}\begin{pmatrix} D_x h_i(x^1, t^1),\ i \in I \\ D_x G(x^1, y^j, t^1),\ j = 1, \ldots, q \end{pmatrix} \leqslant m + q - 1.$$
  Minimal number of equations, i.e. for $\operatorname{rank}(\ldots) = m + q - 1$: $n - m - q + 1$.

Altogether, we obtain $q(n + r + 1)$ equations, and the available dimension is $q(n + r + 1)$, too. Any violation of the nondegeneracy of $(\bar{x}, \bar{t})$ gives rise to an *additional equation* (perhaps using second-order terms in the corresponding multi 2-jet extension). However, in the transversal case, we would satisfy more independent equations than the available number of dimensions; consequently, this will be excluded in the transversal case which is *generic* by virtue of the Multi-jet Transversality Theorem. Next, note that there are countably many possibilities for choosing $q \in \mathbb{N}$ and $B^j \subset B$, $j = 1, \ldots, q$. This proves the theorem taking into account that the bifurcation points from $Y^{(U,V)}$ can be treated analogously. □

REMARK 4.2. The proof of Theorem 4.1 shows that the set

$$\{(H, G, U, V) \in \mathcal{F}_2 \mid |Y_{bp}^{(U,V)}(\bar{t})| + |M_{bp}^{(H,G,U,V)}(\bar{t})| \leqslant 1 \text{ for each } \bar{t} \in \mathbb{R}\}$$

is also generic.

REMARK 4.3. Let us return to the Subtypes 1a and 2a of a nondegenerate point $(\bar{y}, \bar{t}) \in Y_{bp}^{(U,V)}$ and the corresponding possibility of an arbitrary change in the topological structure of $M^{(H,G,U,V)}(t)$ at $t = \bar{t}$ as illustrated in Example 1.1. Note that these two subtypes cannot be excluded generically!

**The set CUSC**

We introduce the following set CUSC (abbreviation for 'compact usc', cf. [13]):

$$\underline{\text{CUSC}} = \left\{ (U,V) \in C^0(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b}) \,\middle|\, \begin{array}{l} \text{For all } \bar{t} \in \mathbb{R} : Y^{(U,V)}(\bar{t}) \text{ is} \\ \text{compact and } t \mapsto Y^{(U,V)}(t) \text{ is usc at } \bar{t}. \end{array} \right\}$$

It is easily seen that $(U,V) \in CUSC$ is equivalent to the fact that there exists for each compact set $T \subset \mathbb{R}$ a compact set $\hat{Y}$ such that $Y^{(U,V)}(t) \subset \hat{Y}$ for all $t \in T$. We already used the upper semicontinuity in Lemma 2.1 and for the local description of $M^{(H,G,U,V)}(t)$ under (RA) in (2.4). Furthermore, we characterized in Section 3 possible changes in the topological structure of $Y^{(U,V)}(t)$ under the assumption that there exist a compact set $\hat{Y} \subset \mathbb{R}^r$ and $t_1, t_2$ with $t_1 < t_2$ such that $Y^{(U,V)}(t) \subset \hat{Y}$ for all $t \in [t_1, t_2]$.

LEMMA 4.4. *The set CUSC is $C_s^0$-open in $C^0$.*

*Proof.* Let $(\bar{U}, \bar{V}) \in CUSC$. We will construct a $C_s^0$-neighbourhood $\vartheta$ of $(\bar{U}, \bar{V})$ such that $\vartheta \subset CUSC$.

*Compactness*. For $(y,t) \in \mathbb{R}^r \times \mathbb{R}$ define the continuous function

$$c(y,t) = \max\{|u_\ell(y,t)|,\ \ell \in A,\ |\min\{0, v_k(y,t)\}|,\ k \in B\}.$$

Then, for $(y,t) \in \{(\mathbb{R}^r \times \mathbb{R}) \setminus Y^{(\bar{U},\bar{V})}\}$ we have $c(y,t) > 0$. For $\gamma > 0$ we obtain the open covering $\{B_\gamma(Y^{(\bar{U},\bar{V})}), (\mathbb{R}^r \times \mathbb{R}) \setminus Y^{(\bar{U},\bar{V})}\}$ of $\mathbb{R}^r \times \mathbb{R}$. By selecting the constant 1 to $B_\gamma(Y^{(\bar{U},\bar{V})})$ as well as $\frac{c(y,t)}{2}$ to $(\mathbb{R}^r \times \mathbb{R}) \setminus Y^{(\bar{U},\bar{V})}$ and using a partition of unity subordinate to this covering we obtain a $C_s^0$-neighbourhood $\vartheta$ of $(\bar{U}, \bar{V})$ such that $Y^{(U,V)} \subset B_\gamma(Y^{(\bar{U},\bar{V})})$ and $Y^{(U,V)}(t)$ is compact for all $t \in \mathbb{R}$ and all $(U,V) \in \vartheta$.

*Upper semicontinuity*. Suppose that there exist $(U,V) \in \vartheta$ and $\tilde{t} \in \mathbb{R}$ such that $t \mapsto Y^{(U,V)}(t)$ is not usc at $\tilde{t}$. Then, there exist a $\bar{\gamma} > 0$ as well as sequences $\{t^\nu\}$ (throughout the paper $\nu$ runs through the set of natural numbers $\mathbb{N}$) and $\{y^\nu\}$ with $t^\nu \to \bar{t}$, $y^\nu \in Y^{(U,V)}(t^\nu)$ and $d(y^\nu, Y^{(U,V)}(\bar{t})) \geqslant \bar{\gamma}$. However, the construction yields $Y^{(U,V)} \subset B_\gamma(Y^{(\bar{U},\bar{V})})$ and, therefore, we have without loss of generality that $y^\nu \to \bar{y}$ with $\bar{y} \in Y^{(U,V)}(\bar{t})$; this is a contradiction. □

Lemma 3.2 provides the following corollary.

COROLLARY 4.5. *The set $\mathcal{F}_1 \cap CUSC$ is $C_s^2$-open/dense in $C^2 \cap CUSC$.*

**The set BAP**

We introduce the set BAP in order to avoid asymptotical effects; in fact, small perturbations of the set $Y^{(U,V)}$ might allow new feasible points to enter 'from infinity'. The following two typical examples illustrate these asymptotical effects:

EXAMPLE 4.6. Let $\bar{t} \in \mathbb{R}$, $Y_1 \subset \mathbb{R}^r$ be a closed subset, $\bar{y} \in \mathbb{R}^r \setminus Y_1$ and

$$Y^{(U,V)}(t) = \begin{cases} Y_1 & t < \bar{t} \\ Y_1 \cup \{\bar{y}\} & t = \bar{t} \end{cases}$$

(cf. Figure 3b). Moreover, assume that there exists a sequence $\{(x^\nu, t^\nu)\} \subset H^{-1}(0)\setminus M^{(H,G,U,V)}$ with $t^\nu > \bar{t}$, $t^\nu \to \bar{t}$, $\|x^\nu\| \to \infty$ and $G(x^\nu, y, t^\nu) \geqslant 0$ for all $y \in Y_1$ (cf. Figure 3a). Then, there exists an arbitrarily small $C_s^1$-perturbation of $(U, V)$ such that for the perturbed vector function $(\tilde{U}, \tilde{V})$ we have $Y^{(\tilde{U},\tilde{V})}(t) = Y_1$ for all $t$ from a neighbourhood of $\bar{t}$ (cf. Figure 3c); hence, we obtain $(x^\nu, t^\nu) \in M^{(H,G,\tilde{U},\tilde{V})}$, i.e. 'some new feasible points arrive from infinity'.

EXAMPLE 4.7. Let $\bar{t} \in \mathbb{R}$, $r = 1$, $y^1 < y^2$ and $Y^{(U,V)}(t) = [y^1, y^2]$ for all $t \in \mathbb{R}$. Furthermore, assume that there exist sequences $(x^\nu, t^\nu) \in H^{-1}(0)\setminus M^{(H,G,U,V)}$ with $t^\nu \to \bar{t}$, $\|x^\nu\| \to \infty$ as well as $\{\bar{y}^\nu\} \subset (y^1, y^2]$ with $\bar{y}^\nu \to y^1$ and $\{y \in [y^1, y^2] \mid G(x^\nu, y, t^\nu) < 0\} = [y^1, \bar{y}^\nu)$ (cf. Figure 4).
After perturbing $(U, V)$ such that $Y^{(\tilde{U},\tilde{V})}(t) = [\hat{y}, y^2]$ for some $\hat{y} \in (y^1, y^2)$ we obtain $(x^\nu, t^\nu) \in M^{(H,G,\tilde{U},\tilde{V})}$, i.e. 'an infinite feasible point becomes finite'.

The set BAP (abbreviation for 'boundedness and properness') is defined as the following subset of $C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^m) \times C^1(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}^r \times \mathbb{R}, \mathbb{R}^{a+b})$: $(H, G, U, V) \in C^1$ belongs to BAP if and only if $(U, V) \in CUSC$ and there exist an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}$ as well as a continuous function $\varepsilon : \mathbb{R} \to (0, 1]$ with:

- cl $M^{(H,G,U,V)} \subset \mathcal{O}$ (where cl denotes the closure),
- for every compact set $T \subset \mathbb{R}$ the set $\mathcal{O} \cap (\mathbb{R}^n \times T)$ is bounded,
- for each $(\bar{x}, \bar{t}) \in H^{-1}(0) \setminus \mathcal{O}$ there exists $\hat{y}(\bar{x}, \bar{t}) \in MF^{(U,V)}(\bar{t})$ such that $G(\bar{x}, y, \bar{t}) < 0$ for all $y \in \text{cl}\, B_{\varepsilon(\bar{t})}(\hat{y}(\bar{x}, \bar{t}))$, and
- $\text{cl}\Big(\bigcup_{(\bar{x},\bar{t})\in H^{-1}(0)\setminus\mathcal{O}} \{(\hat{y}(\bar{x}, \bar{t}), \bar{t})\}\Big) \subset MF^{(U,V)}$.

Obviously, in Example 4.6 we have $(H, G, U, V) \notin BAP$ since $\bar{y} \notin MF^{(U,V)}(\bar{t})$. In Example 4.7 the interval $[y^1, \bar{y}^\nu]$ shrinks to a point for $\nu \to \infty$. Therefore, a continuous function $\varepsilon$ as in the definition of BAP cannot exist which implies $(H, G, U, V) \notin BAP$.

LEMMA 4.8. *The set BAP is $C_s^1$-open in $C^1$.*

*Proof.* The proof is given in 4 steps.

*Step 1.* Assume that $(\bar{H}, \bar{G}, \bar{U}, \bar{V}) \in BAP$. We will construct a $C_s^1$-neighbourhood $\vartheta_1 \times \vartheta_2$ of $(\bar{H}, \bar{G}) \times (\bar{U}, \bar{V})$ such that $\vartheta_1 \times \vartheta_2 \subset BAP$. By Lemma 4.4, assume throughout the proof that the considered $C_s^1$-neighbourhood of $(\bar{U}, \bar{V})$ belongs to CUSC. The construction of $\vartheta_1 \times \vartheta_2$ will be obtained by separate constructions on each stripe $J^\nu = \{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \mid t \in [\nu, \nu + 1]\}$, $\nu \in \mathbb{N}$, where, finally, $\vartheta_1 \times \vartheta_2$ is chosen in such a way that the requirements of these separate constructions are met. Therefore, we focus our consideration on the separate construction of a neighbourhood of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ restricted to $J^0$ (which is denoted by $\vartheta_1^0 \times \vartheta_2^0$). Let $\mathcal{H}_1 = \{(x, t) \in \bar{H}^{-1}(0) \mid t \in [0, 1]\}$ and $\mathcal{B}_1 = \Big\{(y, t) \in \text{cl}\Big(\bigcup_{(\bar{x},\bar{t})\in\bar{H}^{-1}(0)\setminus\mathcal{O}} \{(\hat{y}(\bar{x}, \bar{t}), \bar{t})\}\Big) \,\Big|\, t \in [0, 1]\Big\}$, where, obviously, the latter set is compact.

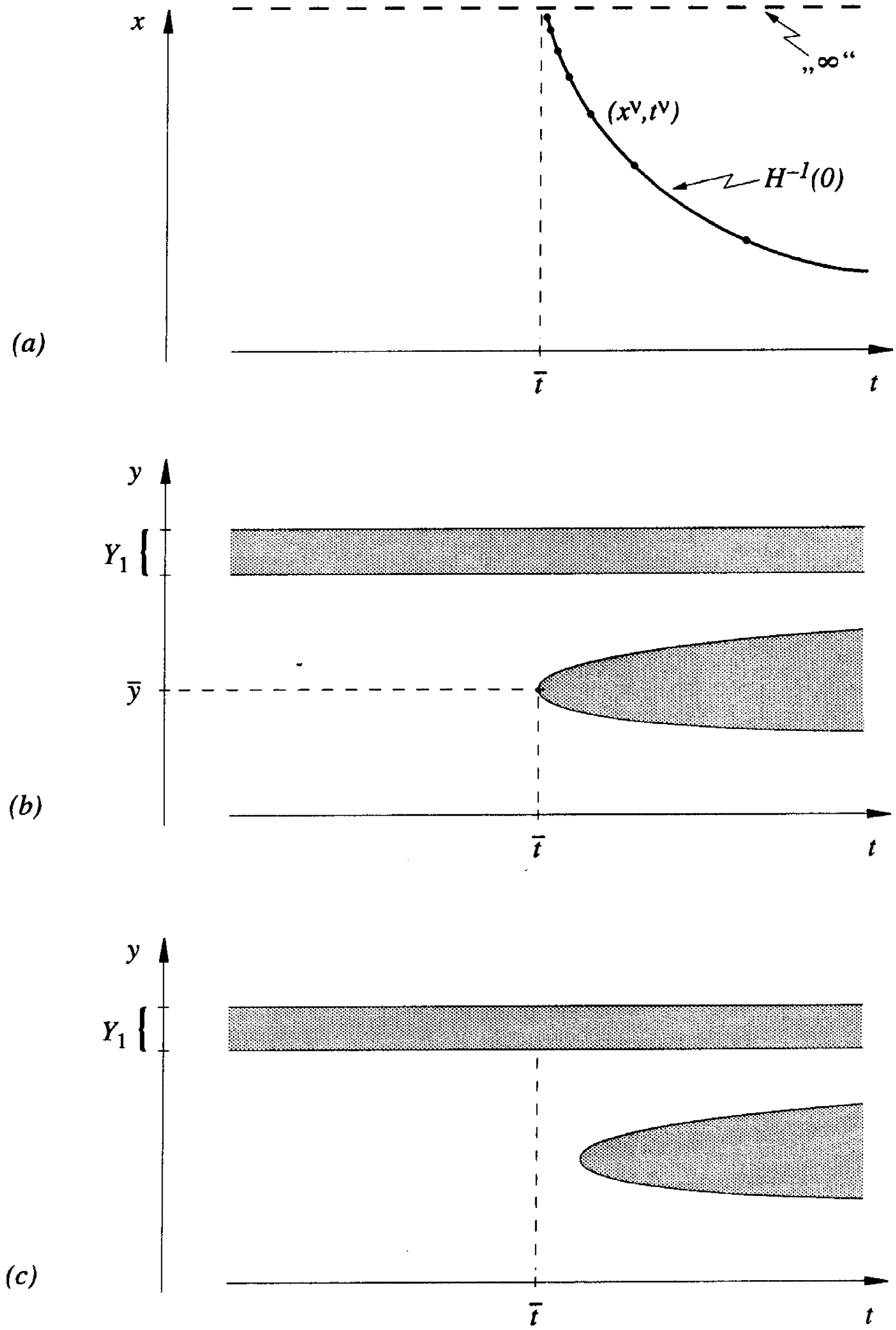


*Figure 3.*

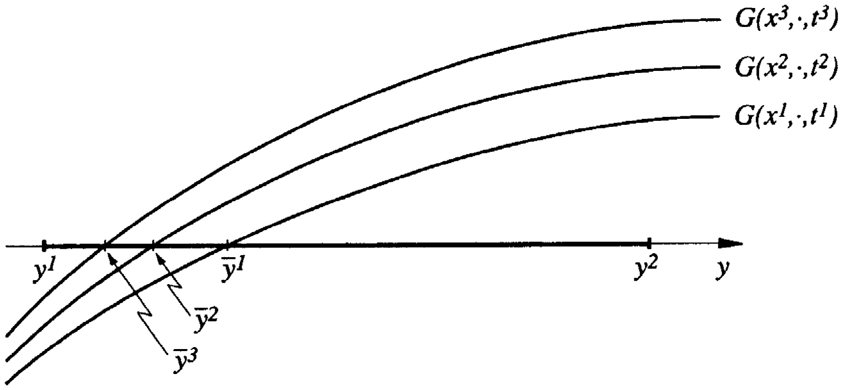


*Figure 4.*

*Step 2.* Lemma 2.1, Lemma 2.2 and the compactness of $\mathcal{B}_1$ imply that there exist $\bar{\varepsilon} \in (0, \frac{1}{2}\min\{\varepsilon(t) \mid t \in [0,1]\})$, and a $C_s^1$-neighbourhood $\vartheta_2^0$ of $(\bar{U}, \bar{V})$ such that for each $(U, V) \in \vartheta_2^0$ and each $(\bar{y}, \bar{t}) \in \mathcal{B}_1$:

- $MF^{(U,V)}(t) \cap B_{\bar{\varepsilon}}(\bar{y}) \neq \emptyset$ for each $t \in \text{cl } B_{\bar{\varepsilon}}(\bar{t})$ and
- $\text{cl } B_{2\bar{\varepsilon}}(\mathcal{B}_1) \cap Y^{(U,V)} \subset MF^{(U,V)}$.

For $(\bar{x}, \bar{t}) \in \mathcal{H}_1 \setminus \mathcal{O}$ choose a neighbourhood $W(\bar{x}, \bar{t})$ of $(\bar{x}, \bar{t})$ such that we have for all $(x, t) \in W(\bar{x}, \bar{t})$:

- $\bar{G}(x, y, t) < \dfrac{\gamma(\bar{x}, \bar{t})}{2}$ for all $y \in \text{cl } B_{\varepsilon(\bar{t})}(\hat{y}(\bar{x}, \bar{t}))$, where $\hat{y}(x, t)$ is as in the definition of BAP and $\gamma(\bar{x}, \bar{t}) = \max\{\bar{G}(\bar{x}, y, \bar{t}) \mid y \in \text{cl } B_{\varepsilon(\bar{t})}(\hat{y}(\bar{x}, \bar{t}))\}$ (obviously, it is $\gamma(\bar{x}, \bar{t}) < 0$), and
- $|t - \bar{t}| < \bar{\varepsilon}$.

Therefore, for $(x, t) \in W(\bar{x}, \bar{t})$ and $(U, V) \in \vartheta_2^0$ we obtain:

- there exists $\tilde{y} \in MF^{(U,V)}(t) \cap B_{\bar{\varepsilon}}(\hat{y}(\bar{x}, \bar{t}))$ and
- $\bar{G}(x, y, t) < \frac{\gamma(\bar{x}, \bar{t})}{2} < 0$ for all $y \in \text{cl } B_{\bar{\varepsilon}}(\tilde{y})$.

*Step 3.* For $(\hat{x}, \hat{t}) \in (\mathbb{R}^n \times [0,1]) \setminus (\mathcal{O} \cup \mathcal{H}_1)$ choose a neighbourhood $W(\hat{x}, \hat{t})$ of $(\hat{x}, \hat{t})$ in such a way that $\sum_{i \in I} |\bar{h}_i(x, t)| > \sum_{i \in I} \frac{|\bar{h}_i(\hat{x}, \hat{t})|}{2}$ $(> 0)$ for all $(x, t) \in W(\hat{x}, \hat{t})$.

*Step 4.* The construction provides a covering $\{\mathcal{O}, \{W(x, t) \mid (x, t) \in (\mathbb{R}^n \times [0,1]) \setminus \mathcal{O}\}\}$ of $\mathbb{R}^n \times [0,1]$ and a covering $\{\mathcal{O} \times \mathbb{R}^r, \{W(x, t) \times \mathbb{R}^r \mid (x, t) \in (\mathbb{R}^n \times [0,1]) \setminus \mathcal{O}\}\}$ of $\mathbb{R}^n \times [0,1] \times \mathbb{R}^r$. Perhaps, after shrinking some $W(x, t)$, choose a subcovering $\{\mathcal{O}, \{W(x^\varrho, t^\varrho) \mid (x^\varrho, t^\varrho) \in (\mathbb{R}^n \times [0,1]) \setminus \mathcal{O}, \varrho \in \tilde{N}\}\}$ of $\mathbb{R}^n \times [0,1]$, where $\tilde{N} \subset \mathbb{N}$, such that for every $\bar{\varrho} \in \tilde{N}$ the set

$$N^{\bar{\varrho}} = \{\varrho \in \tilde{N} \mid W(x^\varrho, t^\varrho) \cap W(x^{\bar{\varrho}}, t^{\bar{\varrho}}) \neq \emptyset\}$$

is finite. For $\bar{\varrho} \in \tilde{N}$ let $N_1^{\bar{\varrho}} = \{\varrho \in N^{\bar{\varrho}} \mid (x^\varrho, t^\varrho) \in \mathcal{H}_1\}$ and $N_2^{\bar{\varrho}} = N^{\bar{\varrho}} \setminus N_1^{\bar{\varrho}}$. By selecting 1 on $\mathcal{O}$ (respectively on $\mathcal{O} \times \mathbb{R}^r$) and

$$\min\left\{\frac{|\gamma(x^{\varrho_1}, t^{\varrho_1})|}{2},\ \varrho_1 \in N_1^{\bar{\varrho}},\ \sum_{i \in I} \frac{|\bar{h}_i(x^{\varrho_2}, t^{\varrho_2})|}{2m},\ \varrho_2 \in N_2^{\bar{\varrho}}\right\}$$

on $W(x^{\bar{\varrho}}, t^{\bar{\varrho}})$ (respectively on $W(x^{\bar{\varrho}}, t^{\bar{\varrho}})\times\mathbb{R}^r$), $\bar{\varrho} \in \tilde{N}$ and using a partition of unity subordinate to this locally finite subcovering we obtain a desired neighbourhood $\vartheta_0^1$ of $(\bar{H}, \bar{G})$. □

## 5. Deformation of $M(t)$

In this section we characterize possible changes in the topological structure of $M^{(H,G,U,V)}(t)$ for varying $t$, where $(H, G, U, V)$ is taken from an appropriate subset that will be characterized in the following theorem.

THEOREM 5.1. *The set*

$$\mathcal{F}_3 = \left\{ (H, G, U, V) \in C^2 \cap BAP \;\middle|\; \begin{array}{l} \bullet \textit{ each } (\bar{x}, \bar{t}) \in M_{bp}^{(H,G,U,V)} \textit{ is nondegenerate,} \\ \bullet \textit{ each } (\bar{y}, \bar{t}) \in Y_{bp}^{(U,V)} \textit{ is nondegenerate,} \\ \bullet\ |Y_{bp}^{(U,V)}(\bar{t})| + |M_{bp}^{(H,G,U,V)}(\bar{t})| \leqslant 1 \textit{ for each} \\ \quad \bar{t} \in \mathbb{R} \textit{ and} \\ \bullet \textit{ there is a set } \mathcal{O} \textit{ as in the definition of BAP} \\ \quad \textit{such that:} \\ \quad \textit{if (EMFCQ) does not hold at an} \\ \quad (\bar{x}, \bar{t}) \in H^{-1}(0) \cap \operatorname{cl} \mathcal{O}, \textit{ then } Y_{bp}^{(U,V)}(\bar{t}) = \emptyset \end{array} \right\}$$

*is $C_s^2$-open/dense in $C^2 \cap BAP$.*

*Proof. Density*. By the proof of Theorem 4.1, $\mathcal{F}_3$ is $C_s^2$-dense in $C^2 \cap BAP$.

*Openess.* Let $(\bar{H}, \bar{G}, \bar{U}, \bar{V}) \in \mathcal{F}_3$ with a corresponding set $\mathcal{O}$. We will construct a $C_s^2$-neighbourhood $\vartheta$ of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ such that $\vartheta \subset \mathcal{F}_3$. By Lemma 3.2 and Lemma 4.8, there exists an open $C_s^2$-neighbourhood $\bar{\vartheta}$ of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ such that for all $(H, G, U, V) \in \bar{\vartheta}$:

- each $(y, t) \in Y_{bp}^{(U,V)}$ is nondegenerate and
- $(H, G, U, V) \in BAP$ with $\operatorname{cl} M^{(H,G,U,V)} \subset \mathcal{O}$.

Without mentioning that again, the following construction of $\vartheta$ will be done in such a way that $\vartheta \subset \bar{\vartheta}$.

We distinguish the following three cases.

*Case 1:* $(\bar{x}, \bar{t}) \in M^{(\bar{H},\bar{G},\bar{U},\bar{V})}$ and (EMFCQ) holds at $\bar{x} \in M^{(\bar{H},\bar{G},\bar{U},\bar{V})}(\bar{t})$. Then, by using continuity arguments, a moment of reflection shows that there exist neighbourhoods $\mathcal{U}(\bar{x}, \bar{t})$ of $(\bar{x}, \bar{t})$ and $\vartheta(\bar{x}, \bar{t})$ of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ such that we obtain for all $(H, G, U, V) \in \vartheta(\bar{x}, \bar{t})$ and all $(x, t) \in \mathcal{U}(\bar{x}, \bar{t}) \cap M^{(H,G,U,V)}$ that (EMFCQ) holds at $x \in M^{(H,G,U,V)}(t)$.

*Case 2:* $(\bar{x}, \bar{t}) \in M_{bp}^{(\bar{H},\bar{G},\bar{U},\bar{V})}$. Obviously, $(\bar{x}, \bar{t})$ is nondegenerate. Then, by using again continuity arguments it is easy to see that there exist neighbourhoods $\mathcal{U}(\bar{x}, \bar{t})$ of $(\bar{x}, \bar{t})$ and $\vartheta(\bar{x}, \bar{t})$ of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ such that we obtain for all $(H, G, U, V) \in \vartheta(\bar{x}, \bar{t})$ and all $(x, t) \in \mathcal{U}(\bar{x}, \bar{t}) \cap M_{bp}^{(H,G,U,V)}$ that $(x, t)$ is nondegenerate.

*Case 3:* $(\bar{x}, \bar{t}) \in \mathrm{cl}\, \mathcal{O} \setminus M^{(\bar{H}, \bar{G}, \bar{U}, \bar{V})}$.

PROPOSITION. *There exist neighbourhoods* $\mathcal{U}(\bar{x}, \bar{t})$ *of* $(\bar{x}, \bar{t})$ *and* $\vartheta(\bar{x}, \bar{t})$ *of* $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ *such that* $(x, t) \notin M_{bp}^{(H,G,U,V)}$ *for all* $(H, G, U, V) \in \vartheta(\bar{x}, \bar{t})$ *and all* $(x, t) \in \mathcal{U}(\bar{x}, \bar{t})$.

*Proof of the Proposition.* Suppose that there are sequences $\{(x^\nu, t^\nu)\}$ and $\{(H^\nu, G^\nu, U^\nu, V^\nu)\}$ satisfying:

- $(x^\nu, t^\nu) \to (\bar{x}, \bar{t})$.
- $(x^\nu, t^\nu) \in M_{bp}^{(H^\nu, G^\nu, U^\nu, V^\nu)}$.
- $h_i^\nu(x^\nu, t^\nu) \to \bar{h}_i(\bar{x}, \bar{t})$, $i \in I$; in particular, $h_i^\nu(x^\nu, t^\nu) = 0$, $i \in I$ implies $(\bar{x}, \bar{t}) \in \bar{H}^{-1}(0) \cap \mathrm{cl}\, \mathcal{O}$ and there is a $\bar{y} \in Y^{(\bar{U}, \bar{V})}(\bar{t})$ with

$$G(\bar{x}, \bar{y}, \bar{t}) < 0. \tag{5.1}$$

- (EMFCQ) does not hold at $x^\nu \in M^{(H^\nu, G^\nu, U^\nu, V^\nu)}(t^\nu)$ and, thus, (EMFCQ) also does not hold at $\bar{x} \in \bar{H}_{\bar{t}}^{-1}(0)$; that implies $Y^{(\bar{U}, \bar{V})}(\bar{t}) = MF^{(\bar{U}, \bar{V})}(\bar{t})$.
- $Y^{(U^\nu, V^\nu)}(t^\nu) \simeq Y^{(\bar{U}, \bar{V})}(\bar{t})$ (which follows from the latter fact and Lemma 2.1) with the homeomorphism

$$\phi^\nu : Y^{(\bar{U}, \bar{V})}(\bar{t}) \longrightarrow Y^{(U^\nu, V^\nu)}(t^\nu),$$

  where $\phi^\nu(\bar{y}) \to \bar{y}$.
- $G^\nu(x^\nu, \phi^\nu(\bar{y}), t^\nu) \to G(\bar{x}, \bar{y}, \bar{t})$. Then, $G^\nu(x^\nu, \phi^\nu(\bar{y}), t^\nu) \geqslant 0$ contradicts (5.1) and Proposition is proved.

From Cases 1, 2 and 3 we obtain a covering $\{\mathcal{U}(\bar{x}, \bar{t}), (\bar{x}, \bar{t}) \in \mathrm{cl}\, \mathcal{O}\}$ of $\mathrm{cl}\, \mathcal{O}$. After selecting $\vartheta(\bar{x}, \bar{t})$ on $\mathcal{U}(\bar{x}, \bar{t})$ for every $(\bar{x}, \bar{t}) \in \mathrm{cl}\, \mathcal{O}$ we get straightforwardly a $C_s^2$-neighbourhood $\tilde{\vartheta}$ of $(\bar{H}, \bar{G}, \bar{U}, \bar{V})$ such that for all $(H, G, U, V) \in \tilde{\vartheta}$ each $(\bar{x}, \bar{t}) \in M_{bp}^{(H,G,U,V)}$ is nondegenerate.

We obtain the desired neighbourhood $\vartheta$ after a—possible—shrinking of $\tilde{\vartheta}$ such that for all $(H, G, U, V) \in \vartheta$ we have

- $Y_{bp}^{(U,V)}(\bar{t}) = \emptyset$ in case that (EMFCQ) does not hold at an $(\bar{x}, \bar{t}) \in H^{-1}(0) \cap \mathrm{cl}\, \mathcal{O}$, and
- $|Y_{bp}^{(U,V)}(\bar{t})| + |M_{bp}^{(H,G,U,V)}(\bar{t})| \leqslant 1$ for each $\bar{t} \in \mathbb{R}$. □

LEMMA 5.2. *Let* $(H, G, U, V) \in C^1$ *be fixed with* $(U, V) \in CUSC$ *as well as* $\bar{t} \in \mathbb{R}$, $\mathcal{W}$ *a neighbourhood of* $\bar{t}$ *and* $\hat{M} \subset \mathbb{R}^n$ *a compact set with* $M(t) \subset \hat{M}$ *for all* $t \in \mathcal{W}$. *Assume that* (*EMFCQ*) *holds at all* $x \in M(\bar{t})$ *and that the mapping*

$$t \mapsto Y(t) \quad \text{is lsc at} \quad \bar{t}. \tag{5.2}$$

*Then, there exists a neighbourhood* $\mathcal{V}$ *of* $\bar{t}$ *such that* $M(t) \simeq M(\bar{t})$ *for all* $t \in \mathcal{V}$.

*Proof.* We distinguish two cases.

*Case 1:* $M(\bar{t}) = \emptyset$. Suppose that there exist sequences $\{t^\nu\}$, $\{x^\nu\}$ with $t^\nu \to \bar{t}$, $x^\nu \in M(t^\nu)$, $M(t^\nu) \subset \hat{M}$ and $x^\nu \to \bar{x}$. Then, there exist a $\bar{y} \in Y(\bar{t})$ with $G(\bar{x}, \bar{y}, \bar{t}) < 0$ and, by (5.2), $y^\nu \in Y(t^\nu)$ with $y^\nu \to \bar{y}$ and $G(x^\nu, y^\nu, t^\nu) < 0$. This is a contradiction.

*Case 2.* $M(\bar{t}) \neq \emptyset$. Since the proof runs in a way analogous to the proof of [3, Theorem B] (in [3, Theorem B] it is assumed that $H \in C^2$; the proof for $H \in C^1$ is given in [8]) we will only recall the main ideas and restrict ourselves to the case $I = \emptyset$. Obviously, it is $\partial M(\bar{t}) \neq \emptyset$. Since (EMFCQ) holds at all $\bar{x} \in M(\bar{t})$, choose for every $\bar{x} \in \partial M(\bar{t})$ a vector $\xi^{\bar{x}} \in \mathbb{R}^n$ with $\|\xi^{\bar{x}}\| < 1$ and $D_x G(\bar{x}, \bar{t}, y)\xi^{\bar{x}} > 0$ for all $y \in Y_0(\bar{x}, \bar{t})$. By continuity, there are neighbourhoods $\mathcal{U}(\bar{x})$ of $\bar{x}$ and $\mathcal{V}(\bar{t})$ of $\bar{t}$ such that for all $(x, t) \in \mathcal{U}(\bar{x}) \times \mathcal{V}(\bar{t})$ we have $D_x G(x, y, t)\xi^{\bar{x}} > 0$ for all $y \in Y_0(x, t)$. By selecting $0 \in \mathbb{R}^n$ on $\mathbb{R}^n \setminus \partial M(\bar{t})$ and $\xi^x$ on $\mathcal{U}(x)$ for $x \in \partial M(\bar{t})$ and by using a partition of unity subordinate to the covering $\{\mathcal{U}(x),\ x \in \partial M(\bar{t}),\ \mathbb{R}^n \setminus \partial M(\bar{t})\}$ of $\mathbb{R}^n$ we obtain a bounded, and, therefore, completely integrable vector field $\xi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with the following properties, where $\psi(x, \tau)$ denotes the flow of $\xi$:

- if $\bar{x} \in \partial M(\bar{t})$, then $D_x G(\bar{x}, y, \bar{t})\xi(\bar{x}) > 0$ for all $y \in Y_0(\bar{x}, \bar{t})$, and
- the set $\mathcal{N}^\varepsilon = \{\psi(x, \tau) \mid x \in \partial M(\bar{t}),\ \tau \in (-\varepsilon, \varepsilon)\}$ is an open neighbourhood of $\partial M(\bar{t})$ for any $\varepsilon > 0$.

Now, let $\varepsilon > 0$ be arbitrarily chosen and fixed. Then, by (5.2), it is easy to verify that there exists a neighbourhood $\mathcal{V}$ of $\bar{t}$ such that for all $t \in \mathcal{V}$:

- $M(t) \subset M(\bar{t}) \cup \mathcal{N}^\varepsilon$,
  (as an example that the lower semicontinuity (5.2) is needed we prove the latter term: Suppose that there are sequences $\{t^\nu\}$, $\{x^\nu\}$ with $t^\nu \to \bar{t}$, $x^\nu \in \Big(M(t^\nu) \setminus (M(\bar{t}) \cup \mathcal{N}^\varepsilon)\Big)$, $x^\nu \to \bar{x}$ and, thus, $\bar{x} \notin M(\bar{t})$. Then, there exist $\bar{y} \in Y(\bar{t})$ with $G(\bar{x}, \bar{y}, \bar{t}) < 0$ and, by (5.2), a sequence $\{y^\nu\}$ with $y^\nu \to \bar{y}$, $y^\nu \in Y(t^\nu)$ and $G(x^\nu, y^\nu, t^\nu) < 0$. This is a contradiction.)
- $M(t) \setminus \mathcal{N}^\varepsilon \subset M(\bar{t})$,
- $\partial M(t) \subset \mathcal{N}^\varepsilon$,
- if $x \in \partial M(t)$, then $D_x G(x, y, t)\xi(x) > 0$ for all $y \in Y_0(x, t)$,
- for each $\bar{x} \in \partial M(\bar{t})$ there exists a uniquely determined integration time $\tau(\bar{x}) \in (-\varepsilon, \varepsilon)$ with $\{\psi(\bar{x}, \tau) \mid \tau \in \mathbb{R}\} \cap \partial M(t) = \{\psi(\bar{x}, \tau(\bar{x}))\}$ and the corresponding function

  $$\mathcal{T} : \bar{x} \in \partial M(\bar{t}) \mapsto \mathcal{T}(\bar{x}) = \psi(\bar{x}, \tau(\bar{x})) \in \partial M(t)$$

  is a homeomorphism.

Then, for each $t \in \mathcal{V}$ the desired homeomorphism, which maps $M(\bar{t})$ onto $M(t)$, is constructed by means of $\mathcal{T}$ and, in particular, it is the identity on $M(\bar{t}) \setminus \mathcal{N}^{2\varepsilon}$. $\square$

## Summary

Finally, we discuss possible changes in the topological structure of $M^{(H,G,U,V)}(t)$ for varying $t$, where $(H, G, U, V)$ is chosen from the set $\mathcal{F}_3$. By Theorem 5.1 and Lemma 4.8, the set $\mathcal{F}_3$ is $C^2_s$-open/dense in $C^2 \cap BAP$ and the latter set is $C^2_s$-open in $C^2$.

The following situation which is assumed in the remainder of this paper for a fixed vector function $(H, G, U, V) \in \mathcal{F}_3$ is typical for mappings from $\mathcal{F}_3$:

- there exist $t_1, t_2$ with $t_1 < t_2$ and a compact set $\hat{M} \subset \mathbb{R}^n$ such that $M(t) \subset \hat{M}$ for all $t \in [t_1, t_2]$, and
- $|(M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2]))| + |(Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]))| \leqslant 1$;
  if $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2])$ is a singleton $\{(\bar{x}, \bar{t})\}$, then $\bar{t} \in (t_1, t_2)$ and $(\bar{x}, \bar{t}) \in M_{bp}$ is nondegenerate as well as
  if $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2])$ is a singleton $\{(\bar{y}, \bar{t})\}$, then $\bar{t} \in (t_1, t_2)$ and $(\bar{y}, \bar{t}) \in Y_{bp}$ is nondegenerate.

Then, we obtain the following five possible cases for a change in the topological structure of $M(t)$:

*Case 1:* $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2]) = \emptyset$ and $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]) = \emptyset$. Then, by Lemma 2.1 and Lemma 5.2, $M(t_1) \simeq M(t_2)$.

*Case 2:* $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2]) = \emptyset$, $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]) = \{(\bar{y}, \bar{t})\}$ and $(\bar{y}, \bar{t})$ is of Type 1a or 2a (cf. Section 3). Then, the change in the topological structure of $M(t)$ might be quite arbitrary (cf. Example 1.1) and, hence, a general description of this case is not possible.

*Case 3:* $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2]) = \emptyset$, $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]) = \{(\bar{y}, \bar{t})\}$ and $(\bar{y}, \bar{t})$ is of Type 1b or 2b. Then, by Lemma 3.5 and Lemma 5.2, $M(t_1) \simeq M(t_2)$.

*Cases 4 and 5:* $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2]) = \{(\bar{x}, \bar{t})\}$ and $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2]) = \emptyset$. Then, (RA) is applicable at $\bar{x} \in M(\bar{t})$ and, therefore, $M(\bar{t})$ can be described locally around $\bar{x}$ as in (2.4) and $(\bar{x}, \bar{t})$ is a nondegenerate bifurcation point of a *finite* problem. As shown in Section 3, $(\bar{x}, \bar{t})$ is either of Type 1 $(\delta_1, \delta_2)$ (Case 4) or of Type 2 $(\delta_1, \delta_2, \delta_3, \delta_4)$ (Case 5), where the Type-numbers $(\delta_1, \delta_2)$ and $(\delta_1, \delta_2, \delta_3, \delta_4)$ are analogously determined. The corresponding changes in the topological structure are given in Theorem 3.3.

We summarize these 5 cases in the following overview:

| Case | $M_{bp} \cap (\mathbb{R}^n \times [t_1, t_2])$ | $Y_{bp} \cap (\mathbb{R}^r \times [t_1, t_2])$ | Change of the topological structure of $M(t)$ |
|---|---|---|---|
| 1 | $\emptyset$ | $\emptyset$ | $M(t_1) \simeq M(t_2)$ |
| 2 | $\emptyset$ | $\{(\bar{y}, \bar{t})\}$ (Type 1a or 2a) | is of global nature and is not controllable in general |
| 3 | $\emptyset$ | $\{(\bar{y}, \bar{t})\}$ (Type 1b or 2b) | $M(t_1) \simeq M(t_2)$ |

| | | | |
|---|---|---|---|
| 4 | $\{(\bar{x}, \bar{t})\}$ (Type 1 $(\delta_1, \delta_2)$) | $\emptyset$ | $M(t_2)$ is obtained from $M(t_1)$ by deleting $S^{\delta_2-1} \times D^{\delta_1-\delta_2}$ and implanting $D^{\delta_2} \times S^{\delta_1-\delta_2-1}$. |
| 5 | $\{(\bar{x}, \bar{t})\}$ (Type 2 $(\delta_1, \delta_2, \delta_3, \delta_4)$) | $\emptyset$ | $\delta_4 = 1$ : $M(t_2)$ is homotopy-equivalent to $M(t_1)$ with $D^{\delta_2}$ attached. $\delta_4 = -1$ : $M(t_1)$ is homotopy-equivalent to $M(t_2)$ with $D^{\delta_2}$ attached. |

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